MATH3210 - SPRING 2024 - SECTION 004

HOMEWORK 5 - *SOLUTIONS*

Problem 1 (40 points)**.** Let $D \subset \mathbb{R}$ be a domain of functions f and g, and $x \in D$. Show *directly* that if f and g are continuous at x, then $f+g$ is continuous at x. *Do not use any theorems, use the ε-δ definition of continuity.*

Solution. Fix $\varepsilon > 0$. Since f and g are continuous at x, there exist constants $\delta_1, \delta_2 > 0$ such that if $|x-y| < \delta_1$, then $|f(x)-f(y)| < \varepsilon/2$ and if $|x-y| < \delta_2$, then $|g(x)-g(y)| < \varepsilon/2$. Set $\delta = \min\{\delta_1, \delta_2\}$. Then if $|x-y| < \delta$, we have the conclusion of continuity for both f and g. In particular:

$$|(f(x)+g(x))-(f(y)+g(y))| \leq |f(x)-f(y)| + |g(x)-g(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

□

Problem 2 (20 points)**.** Prove or find a counterexample: if $I = (a,b) \subset \mathbb{R}$ is an open interval and $f : I \to \mathbb{R}$ is a continuous function, then the image of f is an open interval.

Solution. The statement is false Let $I = (-1,1)$ and f be defined by $f(x) = x^2$. Then the image of f on the domain I is $[0,1)$, which is not a closed interval. □

Problem 3 (40 points)**.** Let f and g be functions defined on the open interval $(-1,1)$, and assume that f and g are continuous at 0. Show that the function h defined by

$$h(x) := \max\{f(x), g(x)\}$$

is continuous at 0.

Solution. We prove continuity in three cases.
Case 1: $f(0) > g(0)$**.** In this case, we claim that there exists $\delta > 0$ such that if $|x| < \delta$, then $h(x) = f(x)$. Since f is continuous at 0, and h agrees with f on a neighborhood of 0, h will also be continuous at 0. We now show the claim. Define $\varepsilon = f(0) - g(0) > 0$. Since both f and g are continuous at 0. there exists $\delta > 0$ such that if $|x| < \delta$, then $|f(x) - f(0)| < \varepsilon/2$ and $|g(x) - g(0)| < \varepsilon/2$. In particular, by the reverse triangle inequality

$$f(x)-g(x) = (f(x)-f(0)) + (f(0)-g(0)) + (g(0)-g(x)) \geq (f(0)-g(0)) - \varepsilon/2 - \varepsilon/2 > \varepsilon - \varepsilon = 0.$$

Case 2: $g(0) > f(0)$**.** This case is completely symmetric with the previous case, interchanging f and g.
Case 3: $g(0) = f(0)$**.** In this case, we prove continuity directly. Let $\varepsilon > 0$. Then since both f and g are continuous at 0, there exists $\delta > 0$ such that if $|x| < \delta$, $|f(x) - f(0)| < \varepsilon$ and $|g(x) - g(0)| < \varepsilon$. Then if $|x| < \delta$, by definition of h, either $h(x) = f(x)$ or $h(x) = g(x)$. Since $h(0) = f(0) = g(0)$, we get that $|h(x) - h(0)| < \varepsilon$. □